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IONIZATION COOLING OF MUONS UNDER A STRONG FOCUSING WITH FIELD OF LONGITUDINAL CURRENT

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Abstract

Considered is the kinetics of ionization cooling of muons in a medium carrying the longitudinal current of high density which provides the particles with a strong focusing reducing to minimum the equilibrium value of transverse beam emitance determined by the multiple scattering. The emitance decrease has an exponential form $\epsilon \cdot e^{-\delta t}$ with a decrement equal to a ratio of mean rate of ionization loss to particle energy when the former is kept constant during the cooling.

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TRANSVERSE EMITANCE

The ionization cooling of muons first discussed by Budker and Skrinsky in 1970-1971 [1] and analyzed by Skrinsky and Parkhomchuk in 1981 [2] is by now the most promising way for compression of transverse emitance of muon beams for projects of $\mu^+ - \mu^-$ colliders being now under wide discussion [3]. The cooling is based on the ionization loss of energy by particle passing through a medium and simultaneous acceleration compensating the longitudinal constituent of lost energy whereas the transverse one is decreasing down the equilibrium value determined by the multiple scattering of particles. To reduce the effect of scattering a strong focusing is to be applied to the beam thus producing a confinement of transverse coordinate of particles which results in diminished increase in beam emitance by the angular spread increase. Such a focusing can be created with a current passing through the cooling medium along the beam axis. Let us investigate the process of cooling with use of kinetic equations method.

For muons the ionization loss of energy and scattering are the only significant processes of particle interaction in cooling medium. If we, as well, leave for a separate consideration below the straggling of ionization losses and single scattering by a large angle, the collision integral in kinetic equation $\frac{df}{dt} = \mathrm{St}\ f$, where f is a distribution function for particles according to coordinates and momenta, is got in a form:

$$\mathrm{St}\ f = -\left(\frac{dE}{dt}\right)_{ion} \frac{E}{p^3c^2}\frac{\partial p^2 f}{\partial p} + \frac{E_k^2 E^2}{4p^4c^4}\Delta_\vartheta f.$$

The scattering contribution in St f - a term containing the Laplacian Δ_ϑ

over conic angle - is taken from [4] with only substitution of pv for E to include the non-relativistic energies. The coefficient E_k^2 is: $E_k^2 = 4\pi Z^2 e^4 n L_c$ with Z standing for nuclear charge of slowing medium, n - for atomic density and L_c - for the Coulomb logarithm $L_c = \ln \frac{\vartheta^2 \max}{\vartheta^2 \min}$. In case the scattering only is taken into account the mean square angle is found as: $\frac{\partial <\vartheta^2>_{scatt}}{\partial t} = \frac{E_k^2}{(pv)^2}$.

Below we will mainly bound our consideration with a range of relativistic energies and small enough angles of particles so that $pv \cong E$ and $\sin\vartheta \cong \vartheta$ citing only shortly some results for non-relativistic case.

The variation of function $P(t, \mathbf{r}, \vartheta, E)\, d^2\mathbf{r}\, d^2\vartheta\, dE$ of particle distribution according to the transverse coordinate $\mathbf{r}$, angle ϑ and energy E with distance t in the slowing medium is described by equation:

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} + \frac{\partial P}{\partial \vartheta}\frac{d\vartheta}{dt} + \left(\frac{\partial P}{\partial E} - \frac{2P}{E} - \frac{\vartheta}{E}\frac{\partial P}{\partial \vartheta}\right)\left(\frac{dE}{dt}\right)_{acc} =$$

$$= \frac{E_k^2}{4E^2}\Delta_\vartheta P - \frac{\partial P}{\partial E}\left(\frac{dE}{dt}\right)_{ion} \qquad (1)$$

With $\frac{d\mathbf{r}}{dt} = \vartheta$ and $\frac{d\vartheta}{dt} = -k\mathbf{r}$, where $k = \frac{e}{pc}\frac{dH}{dr}$ and H is the magnetic field of longitudinal current, one gets from (1) in approximation $p_{||} \cong p \cong E$ (with velocity of light taken equal to unity):

$$\frac{\partial P}{\partial t} + \vartheta\frac{\partial P}{\partial \mathbf{r}} - (\vartheta\frac{\xi_0}{E} + k\mathbf{r})\frac{\partial P}{\partial \vartheta} - 2\xi_0\frac{P}{E} + (\xi_0 - \xi)\frac{\partial P}{\partial E} = \frac{E_k^2}{4E^2}\Delta_\vartheta P \qquad (2)$$

Here ξ_0 stands for acceleration rate: $\left(\frac{dE}{dt}\right)_{acc} = \xi_0$, while ξ for the rate of ionization loss of energy: $\left(\frac{dE}{dt}\right)_{ion} = -\xi$. By cooling at a fixed energy ξ_0 is to be equal to ξ.

Now we multiply the equation (2) by $\vartheta^2, \mathbf{r}\vartheta$ and $\mathbf{r}^2$ in turn and make integrations over all the transverse phase space using a condition of zero values of distribution function P and its derivatives at extreme $\mathbf{r}$ and ϑ. After normalization to number of particles this will give us a system of differential equations for mean values $<\vartheta^2>$, $<\mathbf{r}\vartheta>$ and $<\mathbf{r}^2>$. At $\xi_0 = \xi$ this system is:

$$\frac{\partial <\vartheta^2>}{\partial t} + 2\frac{\xi}{E} <\vartheta^2> + 2k <\mathbf{r}\vartheta> = \frac{E_k^2}{E^2}$$

$$\frac{\partial <\mathbf{r}\vartheta>}{\partial t} + \frac{\xi}{E} <\mathbf{r}\vartheta> + k <\mathbf{r}^2> - <\vartheta^2> = 0 \qquad (3)$$

$$\frac{\partial <\mathbf{r}^2>}{\partial t} - 2 <\mathbf{r}\vartheta> = 0$$

The mean square transverse emitance ϵ is defined through the above mean values as: $\epsilon = \sqrt{<\mathbf{r}^2><\vartheta^2> - <\mathbf{r}\vartheta>^2}$. The differential equation for its definition is:

$$\frac{\partial \epsilon^2}{\partial t} + 2\frac{\xi}{E}\epsilon^2 = \frac{E_k^2}{E^2} <\mathbf{r}^2> \qquad (4)$$

Because an account of logarithmic dependence of E_k^2 on t (through the Coulomb logarithm L_c) makes sufficient complication by integration of system (3) and equation (4) it seems reasonable to neglect this dependence using for L_c definition the effective value of material thickness t_{eff} which will be defined below. In such an approximation the solution for $<\mathbf{r}^2>$ is got in a form:

$$<\mathbf{r}^2> = \frac{2E_k^2}{E^2\delta(\omega^2+\delta^2)}\left\{1 - e^{-\delta t}\left(1 + \frac{\delta^2}{\omega^2}\right) - e^{-\delta t}\frac{\delta}{\omega^2}(\omega\sin\omega t - \delta\cos\omega t)\right\}$$

$$+ <\mathbf{r}^2>_0 \frac{e^{-\delta t}}{2}\left\{1 + \cos\omega t + \frac{\delta^2}{4\omega^2}(1 - \cos\omega t) + \frac{\delta}{\omega}\sin\omega t\right\} \qquad (5)$$

$$+ <\mathbf{r}\vartheta>_0 \frac{e^{-\delta t}}{\omega}\left\{\frac{\delta}{\omega}(1 - \cos\omega t) + 2\sin\omega t\right\} + <\vartheta^2>_0 \frac{2e^{-\delta t}}{\omega^2}(1 - \cos\omega t).$$

Here $\delta = \frac{\xi}{E}, \omega = \sqrt{4k - \delta^2}$, and $<\mathbf{r}^2>_0$, $<\mathbf{r}\vartheta>_0$ and $<\vartheta^2>_0$ are the initial values of $<\mathbf{r}^2>, <\mathbf{r}\vartheta>$ and $<\vartheta^2>$ dependent on initial values of beam emitance ϵ_0 and of lattice functions. The expression for emitance, averaged of the oscillation wave length $2\pi/\omega$, with a proper choice of lattice functions is got in a form:

$$\overline{\epsilon}(t) = \epsilon_0 e^{-\delta t} + \frac{E_k^2}{E^2\delta\sqrt{\omega^2+\delta^2}}(1 - e^{-\delta t}) \qquad (6)$$

So, the transverse emitance reduces with increase in a thickness of cooling material in proportionality with $e^{-\delta t}$. To get the cooling by the order the energy loss is to exceed by 2.3 times the particle energy, for 100 times reduction of ϵ the necessary loss is to be 4.6 E.

The equilibrium emitance at extreme t is equal to:

$$\epsilon_{t\to\infty} = \frac{E_k^2}{E^2\delta\sqrt{\omega^2+\delta^2}} \qquad (7)$$

A significant gain from use of current is, evidently, got by $\omega^2 >> \delta^2$, where the expression (7) is simplified to: $\epsilon_{t\to\infty} = \frac{E_k^2}{2\xi E\sqrt{k}}$. As far as $1/\sqrt{k}$ is about equal to the lattice beta-function β inside the current carrying rod, the above expression can be written in a form: $\epsilon_{t\to\infty} = \frac{E_k^2}{2\xi E}\beta$.

The equilibrium value of mean square angle by $\omega^2 >> \delta^2$ is equal to: $< \vartheta^2 >_{t\to\infty} = \frac{E_k^2}{E^2}\frac{1}{2\delta}$. This defines the effective thickness for multiple scattering as $t_{\text{eff}} \cong \frac{1}{2\delta}$ which can be used for calculation of t-dependent value of the Coulomb logarithm L_c.

From above expressions it follows that the equilibrium emitance is proportional to the ratio of rate of the multiple scattering square angle rise to that of the ionization energy loss. This ratio is ruffly proportional to the nuclear charge of cooling medium that determines the choice of materials for cooler. Among them the lithium looks the most preferable as because of small Z as of being a technological material for high gradient focusing device [5].

Another conclusion consists in a fact that the normalized emitance $pc\epsilon_{t\to\infty}$ depends on energy through the magnitude of $\sqrt{k}$ only, that is in proportionality with $\sqrt{E}$.

Let us evaluate the equilibrium emitance of muons cooled at 2 GeV energy in lithium with use of a current producing the magnetic field gradient of 10 T/mm that is $k = 0.15$ cm^{-2}. With $\xi = 1$ MeV/cm we get $t_{\text{eff}} = 10$m,$L_c \cong 15, E_k^2 = 2.2$ MeV^2 cm^{-1} and $\epsilon_{t\to\infty} = 1.4\ 10^{-3}$ cm.rad. The normalized emitance, $pc\epsilon$, is equal to 2.8 MeV cm.rad.

The equilibrium mean square beam radius, $< r^2 >_{t\to\infty} = \frac{E_k^2}{2k\delta E^2}$, in above considered case is equal to $(0.62\text{ mm})^2$. Thus the magnetic field at the beam surface is about 6T. In fact, within such a size a higher field gradient could be considered. If it is equal to 20 T/mm, k is 0.3 cm $^{-2}$, r.m.s beam radius 0.44 mm and emitance $\epsilon_{t\to\infty} = 1.\ 10^{-3}$ cm. rad. with normalized value $pc\epsilon = 2$ MeV cm.rad. If higher gradient would be available the further reduction of beam emitance is to be achieved in proportionality with square root of gradient rise.

The above mentioned magnitudes of magnetic field gradient could be, evidently, achieved at the final stage of cooling only. Small size of the beam here allows to get them with a moderate current and available magnetic field. At initial stage of cooling the optimum beam radius r is defined through the beam emitance ϵ_0, momentum p and maximum magnetic field H_{max} as: $r^3 = \epsilon_0^2\frac{pc}{eH_{max}}$. With $pc = 2$ GeV, $\epsilon_0 = 0.1$ cm.rad. and $H_{max} = 10$T this means $r \cong 0.9$ cm and field gradient ~1.1 T/mm. By that the current is ~0.45 MA.

A limitation for length of cooler can be put by a probability for single scattering by a large angle. The probability is precisely equal to unity for angles exceeding the value ϑ_1, which defines the mean square angle of multiple scattering as $< \vartheta^2 >_{scatt} = \vartheta_1^2 L_c$, i.e. $\vartheta_1^2 = 4\pi Z^2 e^4 nt/E^2$. This means, by the way, that probability for single scattering over the r.m.s. angle of multiple scattering is equal to $1/L_c$.

Let us conditionally consider the particle being lost when the single scattering angle exceeds the r.m.s. angular spread in the beam. For a beam cooled down to the equilibrium emitance in a length T of material the probability for particle to be scattered over the final r.m.s. angle is: $W(< \vartheta^2 >_{t\to\infty}) = \vartheta_1^2(T)/ < \vartheta^2 >_{t\to\infty} = 2\delta T/L_c$, that is equal to 0.64 if emitance is cooled down by 100 times. But probability for such a scattering is homogeneously distributed along the whole length of cooler where, at the most part, the r.m.s. angular spread is sufficiently larger than the final. Thus, the probability for particle to be lost in a sense defined above is found as $W_{lost} = \frac{\vartheta_1^2(T)}{T}\int_0^T \frac{dt}{<\vartheta^2(t)>}$. This is equal to ~12% in a case of emitance cooled down by 100 times to the equilibrium value at 2 GeV energy.

Another effect which influences to some extend the ionization cooling efficiency is the straggling of ionization loss of energy. Because the losses are compensated by acceleration in average only the straggling will result in additional energy spread and in increase in transverse beam emitance caused by dependence of phase trajectory on particle energy. In homogeneously focusing channel the phase trajectory of particle is described by an ellipse $r^2\sqrt{k} + \frac{\vartheta^2}{\sqrt{k}} = \epsilon$, or $r = \sqrt{\epsilon/\sqrt{k}}\ \sin\phi$ and $\vartheta = \sqrt{\epsilon\sqrt{k}}\ \cos\phi$. Deviation of energy loss from the mean value, i.e. the particle energy deviation ΔE, will result in emitance deviation $\Delta\epsilon = \epsilon\frac{\Delta E}{2E}\cos 2\phi$. The averaging gives $< \Delta\epsilon > = 0$ and $< \Delta\epsilon^2 > = \frac{\epsilon^2 <\Delta E^2>}{8E^2}$, where $< \Delta E^2 >$ is the mean square deviation of energy loss from its mean value. The probability distribution for ΔE is described by Landau curve till the length of cooler does not exceed sufficiently the boundary value $t \cong Q_{max}/G$ where Q_{max} is the maximum energy transfer in a single collision and $G = \frac{2\pi e^4 nZ}{mv^2}$, that is $G = 0.035$ MeV/cm in lithium at $v \cong c$. Integration over Landau curve gives $< \Delta E^2 > \cong GtE$, which becomes of rather large value, comparable with an initial energy spread in muon beam, when transverse emitance is cooled down by 100 times or so. Nevertheless the rate of $< \Delta\epsilon^2 >$ increase with t, $\frac{\partial <\Delta\epsilon^2>}{\partial t} \cong \frac{\epsilon^2 G}{8E}$, is by about 500 times less than the absolute value of corresponding term - the second one in the left hand side - in equation (4).

The above consideration is fully applicable in a range of non-relativistic energies if E in expressions (3) - (7) is replaced by $pv = \frac{p^2c^2}{E}$: $\delta = \frac{\xi E}{p^2c^2}$, $\epsilon_{t\to\infty} = \frac{E_k^2 E}{2\xi p^2 c^2 \sqrt{k}}$ and so on.

If energy is not conserved constant during the cooling we need to add a term $-(\xi - \xi_0)\frac{\partial \epsilon^2}{\partial E}$ to the left hand side of equation (4) as well as the corresponding terms to the equations of system (3). To simplify the solution we reduce (4) by ϵ with account for $< r^2 >= \beta\epsilon \cong \epsilon/\sqrt{k}$. Thus the equation for ϵ reads:

$$\frac{\partial \epsilon}{\partial t} + \frac{\xi_0}{E}\epsilon - (\xi - \xi_0)\frac{\partial \epsilon}{\partial E} = \frac{E_k^2}{2E^2\sqrt{k}} \qquad (8)$$

and its solution is:

$$\epsilon = \epsilon_1 \left(\frac{E}{E_1}\right)^{\nu} + \frac{E_k^2}{E\sqrt{k}(\xi + \xi_0)} \left\{1 - \sqrt{\frac{E}{E_1}}\left(\frac{E}{E_1}\right)^{\nu}\right\} \qquad (9)$$

where $\nu = \frac{\xi_0}{\xi - \xi_0}$, $E = E_1 - (\xi - \xi_0)t$ and index 1 denotes the initial values of particle energy and beam transverse emitance.

When difference between ξ and ξ_0 is small the solutions transforms into:

$$\epsilon \cong \frac{E_k^2}{(\xi + \xi_0)E\sqrt{k}} \left\{1 - \sqrt{\frac{E}{E_1}} \exp\left(-\frac{\xi_0}{E_1}t\right)\right\} + \epsilon_1 \exp\left(-\frac{\xi_0}{E_1}t\right) \qquad (10)$$

which in the limit is similar to (6) for $\omega >> \delta$.

In non-relativistic case the expression (9) transforms into:

$$\epsilon = \epsilon_1 \left(\frac{p}{p_1}\right)^{\nu} + \frac{E_k^2\, p^{+\nu} c^2}{2(\xi_0 - \xi)} \int_{p_1}^{p} \frac{E' dp'}{p'^{\,3+\nu}\sqrt{k'}} \qquad (11)$$

ENERGY SPREAD

The efficient cooling of particle energy spread does not take place in above considered process if there is no specially organized sufficiently strong correlation between the particle energy and mean rate of ionization loss. The natural correlation of proper sign in several GeV energy range is not strong enough. The main contribution to energy dependent part of ξ here makes the term $G \ln \gamma^2$ which provides with a value of longitudinal decrement, $\delta_{||} \cong 2G/E_0$, by about 15 times less the transverse decrement $\delta_\perp = \xi_0/E$.

Some cooling of energy spread can be achieved using a strong dependence of transverse decrement on particle energy. After a large enough t this will result in a coordinate dispersion of mean energy across the beam. When

particles of lowest energies got sufficient reduction of their emitance, a conic diaphragm of heavy material can be inserted in cooling medium thus providing the particles at inner radii of the beam, having the highest energy, with quick loss of energy. This will result in increase in spectral density in lower part of particle spectrum.

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Ионизационное охлаждение мюонов при сильной фокусировке полем прямого тока

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